

Minimal Supersolutions of BSDEs with Lower Semicontinuous Generators

Gregor Heyne*
Michael Kupper*
Christoph Mainberger*



* Humboldt-Universität zu Berlin, Germany

This research was supported by the Deutsche
Forschungsgemeinschaft through the SFB 649 "Economic Risk".

<http://sfb649.wiwi.hu-berlin.de>
ISSN 1860-5664

SFB 649, Humboldt-Universität zu Berlin
Spandauer Straße 1, D-10178 Berlin



Minimal Supersolutions of BSDEs with Lower Semicontinuous Generators*

Gregor Heyne[†] Michael Kupper[‡] Christoph Mainberger[§]

October 19, 2011

We study the existence and uniqueness of minimal supersolutions of backward stochastic differential equations with generators that are jointly lower semicontinuous, bounded below by an affine function of the control variable and satisfy a specific normalization property.

Keywords: Supersolutions of Backward Stochastic Differential Equations; Semimartingale Convergence; Nonlinear Expectations

AMS Subject Classification: Primary 60H20, 60H30

JEL Classification: C61, C65, G11

1 Introduction

On a filtered probability space, the filtration of which is generated by a d -dimensional Brownian motion, we want to give conditions ensuring that the set $\mathcal{A}(\xi, g)$, consisting of all *supersolutions* (Y, Z) of a backward stochastic differential equation with *terminal condition* ξ and *generator* g , has a minimal element. Recall that such a supersolution is a pair (Y, Z) such that, for $0 \leq s \leq t \leq T$,

$$Y_s - \int_s^t g_u(Y_u, Z_u) du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi$$

is satisfied. We call Y the *value process* and Z the *control process* of the supersolution (Y, Z) . We start by considering the process $\mathcal{E}^g(\xi)$ defined as

$$\mathcal{E}_t^g(\xi) = \text{ess inf} \{ Y_t \in L^0(\mathcal{F}_t) : (Y, Z) \in \mathcal{A}(\xi, g) \}, \quad t \in [0, T],$$

and show that, under suitable conditions on the generator and the terminal condition, $\mathcal{E}^g(\xi)$ is already the value process of the unique minimal supersolution, that is, there is a unique control process \hat{Z} such

*We thank Samuel Drapeau, Martin Karliczek and Anthony Réveillac for helpful comments and fruitful discussions.

[†]heyne@math.hu-berlin.de; Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany. Financial support from MATHEON project E.2 is gratefully acknowledged.

[‡]kupper@math.hu-berlin.de; Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany. Financial support from MATHEON project E.11 is gratefully acknowledged.

[§]mainberg@math.hu-berlin.de; Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany. Financial support from the Deutsche Forschungsgemeinschaft via SFB 649 "Economic Risk" is gratefully acknowledged.

that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$. It was recently shown in Drapeau et al. [6] that, if the generator g is jointly lower semicontinuous in y and z , convex in z , monotone in y , and bounded below by an affine function of z , a unique minimal supersolution exists. Their proof was based on convex combinations of a monotone decreasing sequence of càdlàg supermartingales converging pointwise to $\mathcal{E}^g(\xi)$ on the rationals and made use of compactness results for sequences of martingales given in Delbaen and Schachermayer [3]. We follow a different approach to show the existence of a unique minimal supersolution. Starting with the assumption that g is also jointly lower semicontinuous in y and z and *positive* and in addition satisfies a certain *normalization* condition, we find a sequence of supersolutions converging *uniformly* to the càdlàg supermartingale $\mathcal{E}^{g,+}(\xi)$, the right limit of $\mathcal{E}^g(\xi)$. We then use results on convergence of semimartingales given in Barlow and Protter [1] to identify a unique process \hat{Z} such that $(\mathcal{E}^{g,+}(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$. By showing that $\mathcal{E}^{g,+}(\xi)$ always stays below $\mathcal{E}^g(\xi)$, we deduce $\mathcal{E}^{g,+}(\xi) = \mathcal{E}^g(\xi)$ and thus $(\mathcal{E}^g(\xi), \hat{Z})$ is the unique minimal supersolution. Later on, we relax the positivity assumption to that of boundedness below by an affine function of z . Also the normalization condition will be relaxed. Hence both, Drapeau et al. [6] and our work, show the existence of a unique minimal supersolution of BSDEs, but under mutually singular conditions on the generator.

Let us briefly discuss the existing literature on related problems, a broader discussion of which can be found in Drapeau et al. [6]. Nonlinear BSDEs were first introduced in Pardoux and Peng [11]. They gave existence and uniqueness results for the case of Lipschitz generators and square integrable terminal conditions. Kobylanski [10] studies BSDEs with quadratic generators, whereas Delbaen et al. [4] consider superquadratic BSDEs with positive generators that are convex in z and independent of y . Among the first introducing supersolutions of BSDEs were El Karoui et al. [7, Section 2.3]. Further references can also be found in Peng [13], who studies the existence and uniqueness of minimal supersolutions under the assumption of a Lipschitz generator and square integrable terminal conditions. Most recently, Cheridito and Stadje [2] have analyzed existence and stability of supersolutions of BSDEs. They consider terminal conditions which are functionals of the underlying Brownian motion and generators that are convex in z and Lipschitz in y and they work with discrete time approximations of BSDEs. Furthermore, the concept of supersolutions is closely related to Peng's g and G -expectations, see for instance [12, 14, 15], since the mapping $\xi \mapsto \mathcal{E}_0^g(\xi)$ can be seen as a nonlinear expectation. Another related field are stochastic target problems as introduced in Soner and Touzi [20], the solutions of which are obtained by dynamic programming methods and can be characterized as viscosity solutions of second order PDEs.

The remainder of this paper is organized as follows. Setting and notations are specified in Section 2. A precise definition of minimal supersolutions and important structural properties of $\mathcal{E}^g(\xi)$, along with the main existence theorem, can then be found in Sections 3.1 and 3.2. Finally, possible relaxations on the assumptions imposed on the generator, as well as a generalization to the case of arbitrary continuous local martingales, are discussed in Section 3.3.

2 Setting and Notations

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, where the filtration (\mathcal{F}_t) is generated by a d -dimensional Brownian motion W and is assumed to satisfy the usual conditions. For some fixed time horizon $T > 0$ and for all $t \in [0, T]$, the sets of \mathcal{F}_t -measurable random variables are denoted by $L^0(\mathcal{F}_t)$, where random variables are identified in the P -almost sure sense. Let furthermore denote $L^p(\mathcal{F}_t)$ the set of random variables in $L^0(\mathcal{F}_t)$ with finite p -norm, for $p \in [1, +\infty]$. Inequalities and strict inequalities

between any two random variables or processes X^1, X^2 are understood in the P -almost sure or in the $P \otimes dt$ -almost sure sense, respectively. In particular, two càdlàg processes X^1, X^2 satisfying $X^1 = X^2$ are indistinguishable, compare [8, Chapter III]. We denote by \mathcal{T} the set of stopping times with values in $[0, T]$ and hereby call an increasing sequence of stopping times (τ^n) , such that $P[\bigcup_n \{\tau^n = T\}] = 1$, a *localizing sequence of stopping times*. By $\mathcal{S} := \mathcal{S}(\mathbb{R})$ we denote the set of càdlàg progressively measurable processes Y with values in \mathbb{R} . For $p \in [1, +\infty[$, we further denote by \mathcal{H}^p the set of càdlàg local martingales M with finite \mathcal{H}^p -norm on $[0, T]$, that is $\|M\|_{\mathcal{H}^p} := E[\langle M, M \rangle_T^{p/2}]^{1/p} < \infty$. By $\mathcal{L}^p := \mathcal{L}^p(W)$ we denote the set of $\mathbb{R}^{1 \times d}$ -valued, progressively measurable processes Z , such that $\int Z dW \in \mathcal{H}^p$, that is, $\|Z\|_{\mathcal{L}^p} := E[(\int_0^T Z_s^2 ds)^{p/2}]^{1/p}$ is finite. For $Z \in \mathcal{L}^p$, the stochastic integral $(\int_0^t Z_s dW_s)_{t \in [0, T]}$ is well defined, see [16], and is by means of the Burkholder-Davis-Gundy inequality [16, Theorem 48] a continuous martingale. We further denote by $\mathcal{L} := \mathcal{L}(W)$ the set of $\mathbb{R}^{1 \times d}$ -valued, progressively measurable processes Z , such that there exists a localizing sequence of stopping times (τ^n) with $Z1_{[0, \tau^n]} \in \mathcal{L}^1$, for all $n \in \mathbb{N}$. For $Z \in \mathcal{L}$, the stochastic integral $\int Z dW$ is well defined and is a continuous local martingale. Furthermore, for a process X , let X^* denote the following expression $X^* := \sup_{t \in [0, T]} |X_t|$, by which we define the norm $\|X\|_{\mathcal{R}^\infty} := \|X^*\|_{L^\infty}$.

We call a càdlàg semimartingale X a *special semimartingale*, if it can be decomposed into $X = X_0 + M + A$, where M is a local martingale and A a predictable process of finite variation such that $M_0 = A_0 = 0$. Such a decomposition is then unique, compare for instance [16, Chapter III, Theorem 30], and is called the *canonical decomposition* of X .

We will use *normal integrands*, a concept introduced in [18], as generators of BSDEs. Throughout this paper, a normal integrand is a function $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow]-\infty, +\infty]$, such that

- $(y, z) \mapsto g(\omega, t, y, z)$ is jointly lower semicontinuous, for all $(\omega, t) \in \Omega \times [0, T]$;
- $(\omega, t) \mapsto g(\omega, t, y, z)$ is progressively measurable, for all $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$.

For a normal integrand g and progressively measurable processes Y, Z , the process $g(Y, Z)$ is itself progressively measurable and the integral $\int g(Y, Z) ds$ is well defined, P -almost surely, under the assumption that $+\infty - \infty = +\infty$, see [19, Chapter 14.F]. Finally, as long as $g \geq 0$, the lower semicontinuity yields an extended Fatou's lemma, that is,

$$\int \liminf_n g_s(Y_s^n, Z_s^n) ds \leq \liminf_n \int g_s(Y_s^n, Z_s^n) ds,$$

for any sequence $((Y^n, Z^n))$ of progressively measurable processes.

3 Minimal Supersolutions of BSDEs

3.1 First Definitions and Structural Properties

A pair $(Y, Z) \in \mathcal{S} \times \mathcal{L}$ is called a *supersolution* of a BSDE, if, for $0 \leq s \leq t \leq T$, it satisfies

$$Y_s - \int_s^t g_u(Y_u, Z_u) du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi, \quad (3.1)$$

for a normal integrand g as *generator* and a *terminal condition* $\xi \in L^0(\mathcal{F}_T)$. For a supersolution (Y, Z) , we call Y the *value process* and Z its corresponding *control process*. A control process Z is said to be *admissible*, if the continuous local martingale $\int Z dW$ is a supermartingale.

Throughout this paper we say that a generator g is

(POS) positive, if $g(y, z) \geq 0$, for all $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$.

(NOR) normalized, if $g_t(y, 0) = 0$, for all $(t, y) \in [0, T] \times \mathbb{R}$.

We are now interested in the set

$$\mathcal{A}(\xi, g) := \{(Y, Z) \in \mathcal{S} \times \mathcal{L} : Z \text{ is admissible and (3.1) holds}\} \quad (3.2)$$

and the process

$$\mathcal{E}_t^g(\xi) = \text{ess inf} \{Y_t \in L^0(\mathcal{F}_t) : (Y, Z) \in \mathcal{A}(\xi, g)\}, \quad t \in [0, T]. \quad (3.3)$$

For the proof of our main existence theorem we will need some auxiliary results concerning structural properties of $\mathcal{E}^g(\xi)$ and supersolutions (Y, Z) in $\mathcal{A}(\xi, g)$.

Lemma 3.1. *Let g be a generator satisfying (POS). Assume further that $\mathcal{A}(\xi, g) \neq \emptyset$ and $\xi^- \in L^1(\mathcal{F}_T)$. Then $\xi \in L^1(\mathcal{F}_T)$ and, for any $(Y, Z) \in \mathcal{A}(\xi, g)$, the control Z is unique and the value process Y is a supermartingale such that $Y_t \geq E[\xi | \mathcal{F}_t]$. Moreover, the unique canonical decomposition of Y is given by*

$$Y = Y_0 + M - A, \quad (3.4)$$

where $M = \int Z dW$ and A is an increasing, predictable, càdlàg process with $A_0 = 0$.

The proof of Lemma 3.1 can be found in [6, Lemma 3.4].

Proposition 3.2. *Suppose that $\mathcal{A}(\xi, g) \neq \emptyset$ and let $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. If g satisfies (POS), then the process $\mathcal{E}^g(\xi)$ is a supermartingale and*

$$\mathcal{E}_t^g(\xi) \geq \mathcal{E}_t^{g,+}(\xi) := \lim_{\substack{s \downarrow t \\ s \in \mathbb{Q}}} \mathcal{E}_s^g(\xi), \quad \text{for all } t \in [0, T], \quad (3.5)$$

and $\mathcal{E}_T^{g,+}(\xi) := \xi$. In particular, $\mathcal{E}^{g,+}(\xi)$ is a càdlàg supermartingale. Furthermore, the following two pasting properties hold true.

1. Let $(Z^n) \subset \mathcal{L}$ be admissible, $\sigma \in \mathcal{T}$, and $(B_n) \subset \mathcal{F}_\sigma$ a partition of Ω . Then the pasted process $\bar{Z} = Z^1 1_{[0, \sigma]} + \sum_{n \geq 1} Z^n 1_{B_n} 1_{[\sigma, T]}$ is admissible.
2. Let $((Y^n, Z^n)) \subset \mathcal{A}(\xi, g)$, $\sigma \in \mathcal{T}$ and $(B_n) \subset \mathcal{F}_\sigma$ as before. If $Y_{\sigma-}^1 1_{B_n} \geq Y_\sigma^n 1_{B_n}$ holds true for all $n \in \mathbb{N}$, then $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$, where

$$\bar{Y} = Y^1 1_{[0, \sigma]} + \sum_{n \geq 1} Y^n 1_{B_n} 1_{[\sigma, T]} \quad \text{and} \quad \bar{Z} = Z^1 1_{[0, \sigma]} + \sum_{n \geq 1} Z^n 1_{B_n} 1_{[\sigma, T]}.$$

Proof. The proof of the part concerning the process $\mathcal{E}^{g,+}(\xi)$ can be found in [6, Proposition 3.5]. As to the first pasting property, let M^n and \bar{M} denote the stochastic integrals of the Z^n and \bar{Z} , respectively.

First, it follows from $(Z^n) \subset \mathcal{L}$ and from (B_n) being a partition that $\bar{Z} \in \mathcal{L}$ and that $\int_{s \vee \sigma}^{t \vee \sigma} \bar{Z}_u dW_u = \sum 1_{B_n} \int_{s \vee \sigma}^{t \vee \sigma} Z_u^n dW_u$. Now observe that the admissibility of all Z^n yields

$$E[\bar{M}_t - \bar{M}_s | \mathcal{F}_s] = E[M_{(t \wedge \sigma) \vee s}^1 - M_s^1 | \mathcal{F}_s] + E\left[\sum_{n \geq 1} 1_{B_n} E[M_{t \vee \sigma}^n - M_{s \vee \sigma}^n | \mathcal{F}_{s \vee \sigma}] | \mathcal{F}_s\right] \leq 0,$$

for $0 \leq s \leq t \leq T$. Thus, \bar{Z} is admissible. Finally, we can approximate σ from below by some foretelling sequence of stopping times $(\eta_m)^1$, and then show, analogously to [6, Lemma 3.1], that the pair (\bar{Y}, \bar{Z}) satisfies inequality (3.1) and is thus an element of $\mathcal{A}(\xi, g)$. \square

Proposition 3.3. *Let $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots$ be a sequence of stopping times converging to the finite stopping time $\tau^* = \lim_{n \rightarrow \infty} \tau_n$. Further, let (Y^n) be a sequence of càdlàg supermartingales such that $Y_{\tau_n-}^n \geq Y_{\tau_n}^{n+1}$, and which satisfies $Y^n 1_{[\tau_{n-1}, \tau_n[} \geq M 1_{[\tau_{n-1}, \tau_n[}$, where M is a uniformly integrable martingale. Then, for any sequence of stopping times $\sigma_n \in [\tau_{n-1}, \tau_n[$, the limit $Y^\infty := \lim_{n \rightarrow \infty} Y_{\sigma_n}^n$ exists and the process*

$$\bar{Y} := \sum_{n \geq 1} Y^n 1_{[\tau_{n-1}, \tau_n[} + Y^\infty 1_{[\tau^*, \infty[}$$

is a càdlàg supermartingale. Moreover, the limit Y^∞ is independent of the approximating sequence $(Y_{\sigma_n}^n)$ and, if all Y^n are continuous and $Y_{\tau_n}^n = Y_{\tau_n}^{n+1}$, for all $n \in \mathbb{N}$, then \bar{Y} is continuous.

Proof. Note that $(Y_{\sigma_n}^n)$ is a (\mathcal{F}_{σ_n}) -supermartingale. Indeed, if $(\tilde{\eta}_m) \uparrow \tau_n$ is a foretelling sequence of stopping times, then, with $\eta_m := \tilde{\eta}_m \vee \tau_{n-1}$, the family $((Y_{\eta_m}^n)^-)_{m \in \mathbb{N}}$ is uniformly integrable and we obtain

$$\begin{aligned} E[Y_{\sigma_{n+1}}^{n+1} | \mathcal{F}_{\sigma_n}] &= E[E[Y_{\sigma_{n+1}}^{n+1} | \mathcal{F}_{\tau_n}] | \mathcal{F}_{\sigma_n}] \leq E[Y_{\tau_n}^{n+1} | \mathcal{F}_{\sigma_n}] \leq E[Y_{\tau_{n-}}^n | \mathcal{F}_{\sigma_n}] \\ &\leq \liminf_m E[Y_{\eta_m}^n | \mathcal{F}_{\sigma_n}] \leq \liminf_m Y_{\eta_m \wedge \sigma_n}^n = Y_{\sigma_n}^n. \end{aligned}$$

Moreover, $((Y_{\sigma_n}^n)^-)$ is uniformly integrable. Hence, the sequence $(Y_{\sigma_n}^n)$ converges by the supermartingale convergence theorem, see [5, Theorems V.28,29], to some random variable Y^∞ , P -almost surely, and thus \bar{Y} is well-defined. Furthermore, the limit Y^∞ is independent of the approximating sequence $(Y_{\sigma_n}^n)$. Indeed, for any other sequence $(\tilde{\sigma}_n)$ with $\tilde{\sigma}_n \in [\tau_{n-1}, \tau_n[$, the limit $\lim_n Y_{\tilde{\sigma}_n}^n$ exists by the same argumentation. Now $\lim_n Y_{\sigma_n}^n = \lim_n Y_{\tilde{\sigma}_n}^n = Y^\infty$ holds, since the sequence $(\hat{\sigma}_n)$ defined by

$$\hat{\sigma}_n := \begin{cases} \sigma_{\frac{n}{2}} \vee \tilde{\sigma}_{\frac{n}{2}} & , \text{ for } n \text{ even} \\ \sigma_{\frac{n+1}{2}} \wedge \tilde{\sigma}_{\frac{n+1}{2}} & , \text{ for } n \text{ odd} \end{cases}$$

satisfies $\hat{\sigma}_n \in [\tau_{n-1}, \tau_n[$ and $\lim_n Y_{\hat{\sigma}_n}^n$ exists. Thus, all limits must coincide. Next, we show that \bar{Y}^{σ_n} is a supermartingale, for all $n \in \mathbb{N}$. To this end first observe that, for all $0 \leq s \leq t$,

$$\begin{aligned} E[\bar{Y}_t^{\sigma_n} - \bar{Y}_s^{\sigma_n} | \mathcal{F}_s] &= \sum_{k=0}^{n-2} E[E[\bar{Y}_{(\tau_{k+1} \vee s) \wedge t}^{\sigma_n} - \bar{Y}_{(\tau_k \vee s) \wedge t}^{\sigma_n} | \mathcal{F}_{(\tau_k \vee s) \wedge t}] | \mathcal{F}_s] \\ &\quad + E[E[\bar{Y}_{(\sigma_n \vee s) \wedge t}^{\sigma_n} - \bar{Y}_{(\tau_{n-1} \vee s) \wedge t}^{\sigma_n} | \mathcal{F}_{(\tau_{n-1} \vee s) \wedge t}] | \mathcal{F}_s] \\ &\quad + E[E[\bar{Y}_t^{\sigma_n} - \bar{Y}_{(\sigma_n \vee s) \wedge t}^{\sigma_n} | \mathcal{F}_{(\sigma_n \vee s) \wedge t}] | \mathcal{F}_s]. \end{aligned}$$

¹ Such a sequence satisfying $\tilde{\eta}_m < \sigma$, for all $m \in \mathbb{N}$, always exists, since in a Brownian filtration every stopping time is predictable, compare [17, Corollary V.3.3].

Note further that, for each $n \in \mathbb{N}$, the process \bar{Y}^{σ_n} is càdlàg and can only jump downwards, that is, $\bar{Y}_t^{\sigma_n} \geq \bar{Y}_t^{\sigma_n}$, for all $t \in \mathbb{R}$. Observe to this end that, on the one hand, $\bar{Y}_{\tau_k-}^{\sigma_n} = Y_{\tau_k-}^k \geq Y_{\tau_k}^{k+1} = \bar{Y}_{\tau_k}^{\sigma_n}$, for all $0 \leq k \leq n-1$, by assumption, where we assumed $\tau_{k-1} < \tau_k$, without loss of generality. On the other hand, Y^k can only jump downwards. Indeed, as càdlàg supermartingales, all Y^k can be decomposed into $Y^k = Y_0^k + M^k - A^k$, by the Doob-Meyer decomposition theorem [16, Chapter III, Theorem 13], where M^k is a local martingale and A^k a predictable, increasing process with $A_0^k = 0$. Since in a Brownian filtration every local martingale is continuous, the claim follows.

Thus, for all $0 \leq k \leq n-2$, and $(\tilde{\eta}_m) \uparrow \tau_{k+1}$ a foretelling sequence of stopping times, it holds with $\eta_m := \tilde{\eta}_m \vee \tau_k$,

$$\begin{aligned} E[\bar{Y}_{(\tau_{k+1} \vee s) \wedge t}^{\sigma_n} - \bar{Y}_{(\tau_k \vee s) \wedge t}^{\sigma_n} | \mathcal{F}_{(\tau_k \vee s) \wedge t}] &\leq E[\bar{Y}_{((\tau_{k+1}-) \vee s) \wedge t}^{\sigma_n} - \bar{Y}_{(\tau_k \vee s) \wedge t}^{\sigma_n} | \mathcal{F}_{(\tau_k \vee s) \wedge t}] \\ &= E[\liminf_m \bar{Y}_{(\eta_m \vee s) \wedge t}^{\sigma_n} - \bar{Y}_{(\tau_k \vee s) \wedge t}^{\sigma_n} | \mathcal{F}_{(\tau_k \vee s) \wedge t}] \\ &\leq E[\liminf_m Y_{(\eta_m \vee s) \wedge t}^{k+1} | \mathcal{F}_{(\tau_k \vee s) \wedge t}] - Y_{(\tau_k \vee s) \wedge t}^{k+1} \\ &\leq \liminf_m E[Y_{(\eta_m \vee s) \wedge t}^{k+1} | \mathcal{F}_{(\tau_k \vee s) \wedge t}] - Y_{(\tau_k \vee s) \wedge t}^{k+1} \leq 0. \end{aligned}$$

Moreover, $E[\bar{Y}_t^{\sigma_n} - \bar{Y}_{(\sigma_n \vee s) \wedge t}^{\sigma_n} | \mathcal{F}_{(\sigma_n \vee s) \wedge t}] = 0$, as well as

$$E[\bar{Y}_{(\sigma_n \vee s) \wedge t}^{\sigma_n} - \bar{Y}_{(\tau_{n-1} \vee s) \wedge t}^{\sigma_n} | \mathcal{F}_{(\tau_{n-1} \vee s) \wedge t}] \leq E[Y_{(\sigma_n \vee s) \wedge t}^n - Y_{(\tau_{n-1} \vee s) \wedge t}^n | \mathcal{F}_{(\tau_{n-1} \vee s) \wedge t}] \leq 0.$$

Combining this we obtain that $E[\bar{Y}_t^{\sigma_n} | \mathcal{F}_s] \leq \bar{Y}_s^{\sigma_n}$. Furthermore, $\lim_n \bar{Y}_t^{\sigma_n} = \bar{Y}_t$, for all $t \in \mathbb{R}$. Indeed, let us write $\lim_n \bar{Y}_t^{\sigma_n} = \lim_n \bar{Y}_t^{\sigma_n} 1_{\{t < \tau^*\}} + \lim_n \bar{Y}_t^{\sigma_n} 1_{\{t \geq \tau^*\}}$. Then, $\lim_n \bar{Y}_t^{\sigma_n} 1_{\{t \geq \tau^*\}} = \lim_n Y_{\sigma_n}^n 1_{\{t \geq \tau^*\}} = Y^\infty 1_{\{t \geq \tau^*\}} = \bar{Y}_t 1_{\{t \geq \tau^*\}}$ and $\lim_n \bar{Y}_{\sigma_n \wedge t}^{\sigma_n} 1_{\{t < \tau^*\}} = \bar{Y}_t 1_{\{t < \tau^*\}}$. Hence, the claim follows. As a consequence of Fatou's lemma it now holds that

$$E[\bar{Y}_t | \mathcal{F}_s] \leq \liminf_{n \rightarrow \infty} E[\bar{Y}_t^{\sigma_n} | \mathcal{F}_s] \leq \liminf_{n \rightarrow \infty} \bar{Y}_s^{\sigma_n} = \bar{Y}_s,$$

since the family $((\bar{Y}_t^{\sigma_n})^-)$ is uniformly integrable. Hence, \bar{Y} is a supermartingale, which by construction has right-continuous paths and [9, Theorem 1.3.8] then yields that \bar{Y} is even càdlàg. Finally, whenever all Y^n are continuous and $Y_{\tau_n}^n = Y_{\tau_n}^{n+1}$ holds, for all $n \in \mathbb{N}$, the process \bar{Y} is continuous per construction. \square

3.2 Existence and Uniqueness of Minimal Supersolutions

We are now ready to state our main existence result. Possible relaxations of the assumptions (POS) and (NOR) imposed on the generator are discussed in Section 3.3. Note that it is not our focus to investigate conditions assuring the crucial assumption that $\mathcal{A}(\xi, g) \neq \emptyset$. See Drapeau et al. [6] and the references therein for further details.

Theorem 3.4. *Let g be a generator satisfying (POS) and (NOR) and $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. If $\mathcal{A}(\xi, g) \neq \emptyset$, then there exists a unique $\hat{Z} \in \mathcal{L}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$.*

Proof. Step 1: Uniform Limit and Verification. Since $\mathcal{A}(\xi, g) \neq \emptyset$, there exist $(Y^b, Z^b) \in \mathcal{A}(\xi, g)$. From now on we restrict our focus to supersolutions (\bar{Y}, \bar{Z}) in $\mathcal{A}(\xi, g)$ satisfying $\bar{Y}_0 \leq Y_0^b$. Indeed, since we are only interested in minimal supersolutions, we can paste any value process of $(Y, Z) \in \mathcal{A}(\xi, g)$ at $\tau := \inf\{t > 0 : Y_t^b > Y_t\} \wedge T$, such that $\bar{Y} := Y^b 1_{[0, \tau]} + Y 1_{[\tau, T]}$ satisfies $\bar{Y}_0 \leq Y_0^b$, where the corresponding control \bar{Z} is obtained as in Proposition 3.2.

Assume for the beginning that we can find a sequence $((Y^n, Z^n))$ within $\mathcal{A}(\xi, g)$ such that

$$\lim_{n \rightarrow \infty} \|Y^n - \mathcal{E}^{g,+}(\xi)\|_{\mathcal{R}_\infty} = 0. \quad (3.6)$$

Since all Y^n are càdlàg supermartingales, they are, by the Doob-Meyer decomposition theorem, see [16, Chapter III, Theorem 13], special semimartingales with canonical decomposition $Y^n = Y_0^n + M^n - A^n$ as in (3.4). The supermartingale property of all $\int Z^n dW$ and $\xi \in L^1(\mathcal{F}_T)$, compare Lemma 3.1, imply that $E[A_T^n] \leq Y_0^n - E[\xi] \in L^1(\mathcal{F}_T)$. Hence, since each A^n is increasing, $\sup_n E[\int_0^T |dA_s^n|] < \infty$. As (3.6) implies in particular that $\lim_{n \rightarrow \infty} E[(Y^n - \mathcal{E}^{g,+}(\xi))^*] = 0$, it follows from [1, Theorem 1 and Corollary 2] that $\mathcal{E}^{g,+}(\xi)$ is a special semimartingale with canonical decomposition $\mathcal{E}^{g,+}(\xi) = \mathcal{E}_0^{g,+}(\xi) + M - A$ and that

$$\lim_{n \rightarrow \infty} \|M^n - M\|_{\mathcal{H}^1} = 0 \quad , \quad \lim_{n \rightarrow \infty} E[(A^n - A)^*] = 0.$$

The local martingale M is continuous and allows for a representation of the form $M = M_0 + \int \hat{Z} dW$, where $\hat{Z} \in \mathcal{L}$, compare [16, Chapter IV, Theorem 43]. Since

$$E \left[\left(\int_0^T (Z_u^n - \hat{Z}_u)^2 du \right)^{1/2} \right] \xrightarrow{n \rightarrow +\infty} 0,$$

we have that, up to a subsequence, (Z^n) converges $P \otimes dt$ -almost surely to \hat{Z} and $\lim_{n \rightarrow \infty} \int_0^t Z^n dW = \int_0^t \hat{Z} dW$, for all $t \in [0, T]$, P -almost surely, due to the Burkholder-Davis-Gundy inequality. In particular, $\lim_{n \rightarrow \infty} Z^n(\omega) = \hat{Z}(\omega)$, dt -almost surely, for almost all $\omega \in \Omega$.

In order to verify that $(\mathcal{E}^{g,+}(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$, we will use the convergence obtained above. More precisely, for all $0 \leq s \leq t \leq T$, Fatou's lemma together with (3.6) and the lower semicontinuity of the generator yields

$$\begin{aligned} \mathcal{E}_s^{g,+}(\xi) - \int_s^t g_u(\mathcal{E}_u^{g,+}(\xi), \hat{Z}_u) du + \int_s^t \hat{Z}_u dW_u \\ \geq \limsup_n \left(Y_s^n - \int_s^t g_u(Y_u^n, Z_u^n) du + \int_s^t Z_u^n dW_u \right) \geq \limsup_n Y_t^n = \mathcal{E}_t^{g,+}(\xi). \end{aligned}$$

The above, the positivity of g and $\mathcal{E}^{g,+}(\xi) \geq E[\xi | \mathcal{F}_\cdot]$ imply that $\int \hat{Z} dW \geq E[\xi | \mathcal{F}_\cdot] - \mathcal{E}_0^{g,+}(\xi)$. Hence, being bounded from below by a martingale, the continuous local martingale $\int \hat{Z} dW$ is a supermartingale. Thus, \hat{Z} is admissible and $(\mathcal{E}^{g,+}(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$ and therefore, by Lemma 3.1, \hat{Z} is unique. Since we know by Proposition 3.2 that $\mathcal{E}^g(\xi) \geq \mathcal{E}^{g,+}(\xi)$, we deduce that $\mathcal{E}^g(\xi) = \mathcal{E}^{g,+}(\xi)$ by the definition of $\mathcal{E}^g(\xi)$, identifying $(\mathcal{E}^g(\xi), \hat{Z})$ as the unique minimal supersolution.

Step 2: A preorder on $\mathcal{A}(\xi, g)$. As to the existence of $((Y^n, Z^n))$ satisfying (3.6), it is sufficient to show that, for arbitrary $\varepsilon > 0$, we can find a supersolution $(Y^\varepsilon, Z^\varepsilon)$ satisfying

$$\|Y^\varepsilon - \mathcal{E}^{g,+}(\xi)\|_{\mathcal{R}_\infty} \leq \varepsilon. \quad (3.7)$$

We define the following preorder² on $\mathcal{A}(\xi, g)$

$$(Y^1, Z^1) \preceq (Y^2, Z^2) \Leftrightarrow \tau_1 \leq \tau_2 \text{ and } (Y^1, Z^1)1_{[0, \tau_1[} = (Y^2, Z^2)1_{[0, \tau_1[}, \quad (3.8)$$

²Note that, in order to apply Zorn's lemma, we need a partial order instead of just a preorder. To this end we consider equivalence

where, for $i = 1, 2$,

$$\tau_i = \inf \{t \geq 0 : Y_t^i > \mathcal{E}_t^{g,+}(\xi) + \varepsilon\} \wedge T. \quad (3.9)$$

For any totally ordered chain $((Y^i, Z^i))_{i \in I}$ within $\mathcal{A}(\xi, g)$ with corresponding stopping times τ_i , we want to construct an upper bound. If we consider

$$\tau^* = \operatorname{ess\,sup}_{i \in I} \tau_i,$$

we know by the monotonicity of the stopping times that we can find a monotone subsequence (τ_m) of $(\tau_i)_{i \in I}$ such that $\tau^* = \lim_{m \rightarrow \infty} \tau_m$. In particular, τ^* is a stopping time. Furthermore, the structure of the preorder (3.8) yields that the value processes of the supersolutions $((Y^m, Z^m))$ corresponding to the stopping times (τ_m) satisfy

$$Y_{\tau_m}^{m+1} \leq Y_{\tau_m-}^{m+1} = Y_{\tau_m-}^m, \quad \text{for all } m \in \mathbb{N}, \quad (3.10)$$

where the inequality follows from the fact that all Y^m are càdlàg supermartingales, see the proof of Proposition 3.3.

Step 3: A candidate upper bound (\bar{Y}, \bar{Z}) for the chain $((Y^i, Z^i))_{i \in I}$. We construct a candidate upper bound (\bar{Y}, \bar{Z}) for $((Y^i, Z^i))_{i \in I}$ satisfying $P[\tau(\bar{Y}) > \tau^* | \tau^* < T] = 1$, with $\tau(\bar{Y})$ as in (3.9).

To this end, let $(\bar{\sigma}_n)$ be a decreasing sequence of stopping times taking values in the rationals and converging towards τ^* from the right³. Then the stopping times $\hat{\sigma}_n := \bar{\sigma}_n \wedge T$ satisfy $\hat{\sigma}_n > \tau^*$ and $\hat{\sigma}_n \in \mathbb{Q}$, on $\{\tau^* < T\}$, for all n big enough. Let us furthermore define the following stopping time

$$\bar{\tau} := \inf \left\{ t > \tau^* : 1_{\{\tau^* < T\}} |\mathcal{E}_{\tau^*}^{g,+}(\xi) - \mathcal{E}_t^{g,+}(\xi)| > \frac{\varepsilon}{2} \right\} \wedge T. \quad (3.11)$$

Due to the right-continuity of $\mathcal{E}^{g,+}(\xi)$ in τ^* , it follows that $\bar{\tau} > \tau^*$ on $\{\tau^* < T\}$. We now set

$$\sigma_n := \hat{\sigma}_n \wedge \bar{\tau}, \quad \text{for all } n \in \mathbb{N}. \quad (3.12)$$

The above stopping times still satisfy $\lim_{n \rightarrow \infty} \sigma_n = \tau^*$ and $\sigma_n > \tau^*$ on $\{\tau^* < T\}$, for all $n \in \mathbb{N}$. We further define the following sets

$$A_n := \left\{ |\mathcal{E}_{\tau^*}^{g,+}(\xi) - \mathcal{E}_{\sigma_n}^g(\xi)| \vee |\mathcal{E}_{\tau^*}^{g,+}(\xi) - \mathcal{E}_{\sigma_n}^{g,+}(\xi)| < \frac{\varepsilon}{8}, \quad \text{for all } m \geq n \right\}. \quad (3.13)$$

They satisfy $A_n \subset A_{n+1}$ and $\bigcup_n A_n = \Omega$ by definition of the sequence (σ_n) ⁴. Note further that $A_n \in \mathcal{F}_{\sigma_n}$ holds true by construction. By Proposition 3.2 we deduce⁵ that, for each $n \in \mathbb{N}$, there exist $(\tilde{Y}^n, \tilde{Z}^n) \in \mathcal{A}(\xi, g)$ such that

$$\tilde{Y}_{\sigma_n}^n \leq \mathcal{E}_{\sigma_n}^g(\xi) + \frac{\varepsilon}{8}. \quad (3.14)$$

classes of processes. Two supersolutions $(Y^1, Z^1), (Y^2, Z^2) \in \mathcal{A}(\xi, g)$ are said to be equivalent, that is, $(Y^1, Z^1) \sim (Y^2, Z^2)$, if $(Y^1, Z^1) \preceq (Y^2, Z^2)$ and $(Y^2, Z^2) \preceq (Y^1, Z^1)$. This means that they are equal up to their corresponding stopping time $\tau_1 = \tau_2$ as in (3.9). This induces a partial order on the set of equivalence classes and hence the use of Zorn's lemma is justified.

³ Compare [9, Problem 2.24].

⁴ Since on $\{\tau^* < T\}$, $\bar{\tau} > \tau^*$ and $\lim_n \hat{\sigma}_n = \tau^*$ with $\hat{\sigma}_n \in \mathbb{Q}$, it is ensured that there exists some $n_0 \in \mathbb{N}$, depending on ω , such that σ_n takes values in the rationals for all $n \geq n_0$. By definition of $\mathcal{E}^{g,+}(\xi)$ as the right-hand side limit of $\mathcal{E}^g(\xi)$ on the rationals and due to the right-continuity of $\mathcal{E}^{g,+}(\xi)$ in τ^* , both inequalities in the definition of A_n are satisfied for all $n \geq n_0$.

⁵ For a detailed proof, see [6, Proposition 3.2.].

Next we partition Ω into $B_n := A_n \setminus A_{n-1}$, where we set $A_0 := \emptyset$ and $\tau_0 := 0$, and define the candidate upper bound as

$$\bar{Y} = \sum_{m \geq 1} Y^m 1_{[\tau_{m-1}, \tau_m[} + 1_{\{\tau^* < T\}} \sum_{n \geq 1} 1_{B_n} \left[\left(\mathcal{E}_{\tau^*}^{g,+}(\xi) + \frac{\varepsilon}{2} \right) 1_{[\tau^*, \sigma_n[} + \bar{Y}^n 1_{[\sigma_n, T[} \right], \quad \bar{Y}_T = \xi, \quad (3.15)$$

$$\bar{Z} = \sum_{m \geq 1} Z^m 1_{[\tau_{m-1}, \tau_m[} + 1_{\{\tau^* < T\}} \sum_{n \geq 1} \bar{Z}^n 1_{B_n} 1_{[\sigma_n, T]}. \quad (3.16)$$

Step 4: Verification of $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$. By verifying that the pair (\bar{Y}, \bar{Z}) is an element of $\mathcal{A}(\xi, g)$, we identify (\bar{Y}, \bar{Z}) as an upper bound for the chain $((Y^i, Z^i))_{i \in I}$. Even more, $P[\tau(\bar{Y}) > \tau^* \mid \tau^* < T] = 1$ holds true, since, on the set B_n , we have $\bar{Y}_t = \mathcal{E}_{\tau^*}^{g,+}(\xi) + \frac{\varepsilon}{2} \leq \mathcal{E}_t^{g,+}(\xi) + \varepsilon$, for all $t \in [\tau^*, \sigma_n[$, due to the definition of $\bar{\tau}$ in (3.11).

Step 4a: The value process \bar{Y} is an element of \mathcal{S} . By construction, the only thing to show is that \bar{Y}_{τ^*-} , the left limit at τ^* , exists. This follows from Proposition 3.3, since, by means of $((Y^m, Z^m)) \subset \mathcal{A}(\xi, g)$ and $\xi \in L^1(\mathcal{F}_T)$, all Y^m are càdlàg supermartingales, see Lemma 3.1, which are bounded from below by a uniformly integrable martingale, more precisely $Y^m \geq E[\xi \mid \mathcal{F}_\cdot]$, for all $m \in \mathbb{N}$, and satisfy (3.10).

Step 4b: The control process \bar{Z} is an element of \mathcal{L} and admissible. We proceed by defining, for each $n \in \mathbb{N}$, the processes $\bar{Z}^n := \sum_{m=1}^n Z^m 1_{[\tau_{m-1}, \tau_m[} = \bar{Z} 1_{[0, \tau_n[} = Z^n 1_{[0, \tau_n[}$ and $N^n := \int \bar{Z}^n dW = \int Z^n 1_{[0, \tau_n[} dW$, where the equalities follow from (3.8). Observe that $N^{n+1} 1_{[0, \tau_n[} = N^n 1_{[0, \tau_n[}$, for all $n \in \mathbb{N}$, and that (POS), (3.1) and the supermartingale property of $\int Z^n dW$ imply

$$N^n 1_{[\tau_{n-1}, \tau_n[} \geq 1_{[\tau_{n-1}, \tau_n[} (-E[\xi^- \mid \mathcal{F}_\cdot] - Y_0^b). \quad (3.17)$$

By means of (3.17) and since $\xi^- \in L^1(\mathcal{F}_T)$, with $N^\infty := \lim_n N_{\tau_{n-1}}^n$, the process

$$N = \sum_{n \geq 1} N^n 1_{[\tau_{n-1}, \tau_n[} + 1_{[\tau^*, T]} N^\infty$$

is a well-defined continuous supermartingale due to Proposition 3.3. Hence we may define a localizing sequence by setting $\kappa_n := \inf\{t \geq 0 : |N_t| > n\} \wedge T$ and deduce that N is a continuous local martingale, because N^{κ_n} is a uniformly integrable martingale, for all $n \in \mathbb{N}$. Indeed, for each $n \in \mathbb{N}$ and $m \in \mathbb{N}$, the process $(N^m)^{\kappa_n}$, being a bounded stochastic integral, is a martingale. Moreover, the family $(N_{\kappa_n \wedge t}^m)_{m \in \mathbb{N}}$ is uniformly integrable and $N_{\kappa_n \wedge t} = \lim_m N_{\kappa_n \wedge t}^m$, for all $t \in [0, T]$. Consequently, $E[N_{\kappa_n}^{\kappa_n} \mid \mathcal{F}_s] = \lim_m E[N_{\kappa_n \wedge t}^m \mid \mathcal{F}_s] = \lim_m N_{\kappa_n \wedge s}^m = N_s^{\kappa_n}$, for all $0 \leq s \leq t \leq T$, and the claim follows. Since the quadratic variation of a continuous local martingale is continuous and unique, see [9, page 36], we obtain

$$\int_0^{\tau^*} \bar{Z}_u^2 du = \lim_n \int_0^{\kappa_n \wedge \tau^*} \bar{Z}_u^2 du = \lim_n \langle N \rangle_{\kappa_n \wedge \tau^*} = \langle N \rangle_{\tau^*} < \infty.$$

Observe that $\sigma := \sum_{n \geq 1} 1_{B_n} \sigma_n$ is an element of \mathcal{T} . Indeed, $\{\sigma \leq t\} = \bigcup_{n \geq 1} (B_n \cap \{\sigma \leq t\}) = \bigcup_{n \geq 1} (B_n \cap \{\sigma_n \leq t\}) \in \mathcal{F}_t$, for all $t \in [0, T]$, since $B_n \in \mathcal{F}_{\sigma_n}$. From $\bar{Z} 1_{[\tau^*, \sigma]} = 0$ we get that

$$\int_0^T \bar{Z}_u^2 du = \langle N \rangle_{\tau^*} + 1_{\{\tau^* < T\}} \sum_{n \geq 1} 1_{B_n} \int_{\sigma}^T (\bar{Z}_u^n)^2 du < \infty,$$

since $(\tilde{Z}^n) \subset \mathcal{L}$. Hence we conclude that $\bar{Z} \in \mathcal{L}$. As for the supermartingale property of $\int \bar{Z} dW$, observe that

$$\int_0^{t \wedge \tau^*} \bar{Z}_u dW_u = \lim_{n \rightarrow \infty} \int_0^{t \wedge \tau_n} Z_u^n dW_u \geq \lim_{n \rightarrow \infty} -E[\xi^- | \mathcal{F}_{t \wedge \tau_n}] - Y_0^b = -E[\xi^- | \mathcal{F}_{t \wedge \tau^*}] - Y_0^b,$$

where the inequality follows from (3.1) and (POS). Being bounded from below by a martingale, we deduce by Fatou's lemma that $\bar{Z}1_{[0, \tau^*]}$ is admissible. Since $\bar{Z}1_{[\tau^*, \sigma]} = 0$ and all \tilde{Z}^n are admissible, it follows from Proposition 3.2 that \bar{Z} is indeed admissible.

Step 4c: The pair (\bar{Y}, \bar{Z}) is a supersolution. Finally, showing that (\bar{Y}, \bar{Z}) satisfy (3.1) identifies (\bar{Y}, \bar{Z}) as an element of $\mathcal{A}(\xi, g)$. Observe first that, for all $0 \leq s \leq t \leq T$ and all $m \in \mathbb{N}$, the expression $\bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^t \bar{Z}_u dW_u$ can be written as

$$\begin{aligned} \bar{Y}_s - \int_s^{(\tau_m \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^{(\tau_m \vee s) \wedge t} \bar{Z}_u dW_u \\ - \int_{(\tau_m \vee s) \wedge t}^{(\tau^* \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du + \int_{(\tau_m \vee s) \wedge t}^{(\tau^* \vee s) \wedge t} \bar{Z}_u dW_u - \int_{(\tau^* \vee s) \wedge t}^{(\sigma \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du \\ + \int_{(\tau^* \vee s) \wedge t}^{(\sigma \vee s) \wedge t} \bar{Z}_u dW_u - \int_{(\sigma \vee s) \wedge t}^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_{(\sigma \vee s) \wedge t}^t \bar{Z}_u dW_u. \end{aligned} \quad (3.18)$$

Now, we have that

$$\bar{Y}_s - \int_s^{(\tau_m \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^{(\tau_m \vee s) \wedge t} \bar{Z}_u dW_u \geq \bar{Y}_{(\tau_m \vee s) \wedge t}, \quad (3.19)$$

by Proposition 3.2, since $((Y^m, Z^m)) \subset \mathcal{A}(\xi, g)$ and $Y_{\tau_m-}^m \geq Y_{\tau_m}^{m+1}$, for all $m \in \mathbb{N}$, due to (3.10). By letting m tend to infinity and noting that

$$\lim_{m \rightarrow \infty} \int_{(\tau_m \vee s) \wedge t}^{(\tau^* \vee s) \wedge t} \bar{Z}_u dW_u = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} \int_{(\tau_m \vee s) \wedge t}^{(\tau^* \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du = 0,$$

(3.18) and (3.19) yield that

$$\begin{aligned} \bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^t \bar{Z}_u dW_u \\ \geq \bar{Y}_{((\tau^* -) \vee s) \wedge t} - \int_{(\tau^* \vee s) \wedge t}^{(\sigma \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du + \int_{(\tau^* \vee s) \wedge t}^{(\sigma \vee s) \wedge t} \bar{Z}_u dW_u \\ - \int_{(\sigma \vee s) \wedge t}^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_{(\sigma \vee s) \wedge t}^t \bar{Z}_u dW_u. \end{aligned} \quad (3.20)$$

We now use that \bar{Y} can only jump downwards at τ^* . Indeed, since \bar{Y} is càdlàg, in particular \bar{Y}_{τ^*-} , the left limit at τ^* , exists and is unique, P -almost surely. Furthermore, $\lim_{m \rightarrow \infty} \bar{Y}_{\tau_m-} = \bar{Y}_{\tau^*-}$ and thus

$$\bar{Y}_{\tau^*-} = \lim_m \bar{Y}_{\tau_m-} = \lim_m Y_{\tau_m-}^m \geq \lim_m \mathcal{E}_{\tau_m-}^{g,+}(\xi) + \varepsilon = \mathcal{E}_{\tau^*-}^{g,+}(\xi) + \varepsilon \geq \mathcal{E}_{\tau^*}^{g,+}(\xi) + \varepsilon > \bar{Y}_{\tau^*}.$$

The second inequality holds, since the càdlàg supermartingale $\mathcal{E}^{g,+}(\xi)$ can only jump downwards, see the proof of Proposition 3.3. Hence, (3.20) can be further estimated by

$$\bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^t \bar{Z}_u dW_u \geq \bar{Y}_{(\tau^* \vee s) \wedge t} - \int_{(\sigma \vee s) \wedge t}^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_{(\sigma \vee s) \wedge t}^t \bar{Z}_u dW_u, \quad (3.21)$$

where we used that

$$\int_{(\tau^* \vee s) \wedge t}^{(\sigma \vee s) \wedge t} g_u(\bar{Y}_u, \bar{Z}_u) du = \int_{(\tau^* \vee s) \wedge t}^{(\sigma \vee s) \wedge t} \bar{Z}_u dW_u = 0,$$

due to (3.16), the definition of σ and (NOR). Now observe that $\bar{Y}_{(\tau^* \vee s) \wedge t} \geq \bar{Y}_{(\sigma \vee s) \wedge t}$, since $\bar{Y} 1_{[\tau^*, \sigma[} = (\mathcal{E}_{\tau^*}^{g,+}(\xi) + \frac{\varepsilon}{2}) 1_{[\tau^*, \sigma[}$ and \bar{Y} can only jump downwards at σ . Indeed, on the set B_n , by means of (3.15), (3.13) and (3.14) holds

$$\begin{aligned} \bar{Y}_{\sigma_n-} &= \mathcal{E}_{\tau^*}^{g,+}(\xi) + \frac{\varepsilon}{2} = \mathcal{E}_{\tau^*}^{g,+}(\xi) - \mathcal{E}_{\sigma_n}^g(\xi) + \mathcal{E}_{\sigma_n}^g(\xi) + \frac{\varepsilon}{2} \\ &\geq -\frac{\varepsilon}{8} + \mathcal{E}_{\sigma_n}^g(\xi) + \frac{\varepsilon}{2} \geq \tilde{Y}_{\sigma_n}^n - \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \tilde{Y}_{\sigma_n}^n = \bar{Y}_{\sigma_n}. \end{aligned}$$

Consequently,

$$\begin{aligned} \bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_s^t \bar{Z}_u dW_u \\ \geq \bar{Y}_{(\sigma \vee s) \wedge t} - \int_{(\sigma \vee s) \wedge t}^t g_u(\bar{Y}_u, \bar{Z}_u) du + \int_{(\sigma \vee s) \wedge t}^t \bar{Z}_u dW_u \geq \bar{Y}_t, \quad (3.22) \end{aligned}$$

where the second inequality in (3.22) follows from $((\tilde{Y}^n, \tilde{Z}^n)) \subset \mathcal{A}(\xi, g)$ and Proposition 3.2.

Step 5: The maximal element (Y^M, Z^M) . By Zorn's lemma, there exists a maximal element (Y^M, Z^M) in $\mathcal{A}(\xi, g)$ with respect to the preorder (3.8), satisfying, without loss of generality, $Y_T^M = \xi$. By showing that the corresponding stopping time satisfies $\tau^M = T$, we have obtained a supersolution (Y^M, Z^M) satisfying $\|Y^M - \mathcal{E}^{g,+}(\xi)\|_{\mathcal{R}^\infty} \leq \varepsilon$, due to the definition of τ^M in analogy to (3.9). Thus, choosing $Y^M = Y^\varepsilon$ in (3.7) would finish our proof.

But on $\{\tau^M < T\}$ we could consider the chain consisting only of (Y^M, Z^M) and, analogously to (3.15) and (3.16), construct an upper bound (\bar{Y}, \bar{Z}) , with corresponding stopping time $\tau(\bar{Y})$ as in (3.9), satisfying $P[\tau(\bar{Y}) > \tau^M \mid \tau^M < T] = 1$. This yields $P[\tau^M < T] \leq P[\tau(\bar{Y}) > \tau^M] = 0$, due to the maximality of τ^M . Hence we deduce that $\tau^M = T$. \square

The techniques used in the proof of Theorem 3.4 show that $\mathcal{A}(\xi, g)$ exhibits a certain closedness under monotone limits of decreasing supersolutions.

Theorem 3.5. *Let g be a generator satisfying (POS) and (NOR) and $\xi \in L^0(\mathcal{F}_T)$ a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. Let furthermore $((Y^n, Z^n))$ be a decreasing sequence within $\mathcal{A}(\xi, g)$ with pointwise limit $\bar{Y}_t := \lim_n Y_t^n$, for $t \in [0, T]$. Then \bar{Y} is a supermartingale and it holds*

$$\bar{Y}_t \geq \hat{Y}_t := \lim_{\substack{s \downarrow t \\ s \in \mathbb{Q}}} \bar{Y}_s \quad \text{for all } t \in [0, T]. \quad (3.23)$$

Moreover, with $\hat{Y}_T := \xi$, there is a sequence $((\tilde{Y}^n, \tilde{Z}^n)) \subset \mathcal{A}(\xi, g)$ such that $\lim_n \|\tilde{Y}^n - \hat{Y}\|_{\mathcal{R}^\infty} = 0$, and a unique control $\hat{Z} \in \mathcal{L}$ such that $(\hat{Y}, \hat{Z}) \in \mathcal{A}(\xi, g)$.

Proof. First, \bar{Y} is a supermartingale by monotone convergence. Inequality (3.23) is then proved analogously to (3.5) as in [6, Proposition 3.5]. The rest follows by adapting all steps in the proof of Theorem 3.4 and replacing $\mathcal{E}^{g,+}(\xi)$ by \hat{Y} . \square

In a next step, we turn our focus to the question whether it is possible to find a minimal supersolution within $\mathcal{A}(\xi, g)$, the associated control process Z of which belongs to \mathcal{L}^1 and therefore $\int Z dW$ constitutes a true martingale instead of only a supermartingale. To this end we consider the following subset of $\mathcal{A}(\xi, g)$

$$\mathcal{A}^1(\xi, g) := \{(Y, Z) \in \mathcal{A}(\xi, g) : Z \in \mathcal{L}^1\}. \quad (3.24)$$

By imposing stronger assumptions on the terminal condition ξ , the next theorem yields the existence of a unique minimal supersolution in $\mathcal{A}^1(\xi, g)$.

Theorem 3.6. *Assume that the generator g satisfies (POS) and (NOR) and let $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $(E[\xi^- | \mathcal{F}_\cdot])^* \in L^1(\mathcal{F}_T)$. If $\mathcal{A}^1(\xi, g) \neq \emptyset$, then there exists a unique \hat{Z} such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}^1(\xi, g)$.*

Proof. $\mathcal{A}^1(\xi, g) \neq \emptyset$ yields that $\mathcal{A}(\xi, g) \neq \emptyset$, because $\mathcal{A}^1(\xi, g) \subseteq \mathcal{A}(\xi, g)$. Also, from $(E[\xi^- | \mathcal{F}_\cdot])^*_T \in L^1(\mathcal{F}_T)$ we deduce that $\xi^- \in L^1(\mathcal{F}_T)$. Hence, Theorem 3.4 yields the existence of a unique control \hat{Z} , such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$. Verifying that $\hat{Z} \in \mathcal{L}^1$ is done as in [6, Theorem 4.5]. \square

3.3 Relaxations of the Conditions (NOR) and (POS)

In this section we discuss possible relaxations of the conditions (NOR) and (POS) imposed on the generator throughout Sections 3.1 and 3.2.

First, we want to replace (NOR) by the weaker assumption (NOR'). We say that a generator g satisfies

(NOR') if, for all $\tau \in \mathcal{T}$, there exists some stopping time $\delta > \tau$ such that the stochastic differential equation

$$dy_s = g_s(y_s, 0)ds, \quad y_\tau = \mathcal{E}_\tau^{g,+}(\xi) + \frac{\varepsilon}{2} \quad (3.25)$$

admits a solution on $[\tau, \delta]$.

By this we obtain the following corollary to Theorem 3.4.

Corollary 3.7. *Let g be a generator satisfying (POS) and (NOR') and $\xi \in L^0(\mathcal{F}_T)$ a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. If $\mathcal{A}(\xi, g) \neq \emptyset$, then there exists a unique $\hat{Z} \in \mathcal{L}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$.*

Proof. We will proceed along the lines of the proof of Theorem 3.4 with the focus on the required alterations.

The first difficulty lies in the pasting at the stopping time τ^* within the definition of (\bar{Y}, \bar{Z}) in (3.15) and (3.16). Instead of extending by a constant function, we concatenate the value process at τ^* with the solution of the SDE (3.25), started at $y_{\tau^*} = \mathcal{E}_{\tau^*}^{g,+}(\xi) + \frac{\varepsilon}{2}$ and denoted by y . We emphasize that the zero control is maintained.

Furthermore, we have to introduce an additional stopping time and adjust $\bar{\tau}$ defined in (3.11), in order to ensure that our constructed value process does not leave the ε -neighborhood of $\mathcal{E}^{g,+}(\xi)$. We define

$$\kappa := \inf\{t > \tau^* : 1_{\{\tau^* < T\}} \int_{\tau^*}^t g_s(y_s, 0) ds > \frac{\varepsilon}{6}\} \wedge \delta \quad (3.26)$$

$$\text{and } \bar{\tau} := \inf\{t > \tau^* : 1_{\{\tau^* < T\}} |\mathcal{E}_{\tau^*}^{g,+}(\xi) - \mathcal{E}_t^{g,+}(\xi)| > \frac{\varepsilon}{6}\} \wedge T \quad (3.27)$$

and use $\bar{\kappa} := \kappa \wedge \bar{\tau}$ within the definition of the sequence (σ_n) in analogy to (3.12), that is, $\sigma_n = \hat{\sigma}_n \wedge \bar{\kappa}$, for all $n \in \mathbb{N}$. As before, we set $\sigma := \sum_{n \geq 1} 1_{B_n} \sigma_n$.

The pasting in (3.15) and (3.16) is done analogously to the proof of Theorem 3.4, but now with the distinction that $\bar{Y} 1_{[\tau^*, \sigma]} = y_{\tau^*} + 1_{[\tau^*, \sigma]} \int_{\tau^*}^{\cdot} g_s(y_s, 0) ds$. The definition of the stopping times κ , $\bar{\tau}$ and σ implies that, on the set B_n , we have $\bar{Y}_t \leq \mathcal{E}_t^{g,+}(\xi) + \varepsilon$, for all $t \in [\tau^*, \sigma_n]$. Indeed, observe that, for $t \in [\tau^*, \sigma_n]$,

$$\begin{aligned} \bar{Y}_t &= \mathcal{E}_{\tau^*}^{g,+}(\xi) + \frac{\varepsilon}{2} + \int_{\tau^*}^t g_s(y_s, 0) ds \leq \mathcal{E}_{\tau^*}^{g,+}(\xi) + \frac{2\varepsilon}{3} \\ &= \mathcal{E}_{\tau^*}^{g,+}(\xi) - \mathcal{E}_t^{g,+}(\xi) + \mathcal{E}_t^{g,+}(\xi) + \frac{2\varepsilon}{3} \leq \mathcal{E}_t^{g,+}(\xi) + \frac{\varepsilon}{6} + \frac{2\varepsilon}{3} < \mathcal{E}_t^{g,+}(\xi) + \varepsilon, \end{aligned}$$

by means of (3.26) and (3.27), together with the definition of σ_n . Furthermore, on the set B_n ,

$$\bar{Y}_{\sigma_n-} = \mathcal{E}_{\tau^*}^{g,+}(\xi) + \frac{\varepsilon}{2} + \int_{\tau^*}^{\sigma_n} g_s(y_s, 0) ds \geq \mathcal{E}_{\tau^*}^{g,+}(\xi) + \frac{\varepsilon}{2} \geq \bar{Y}_{\sigma_n}. \quad (3.28)$$

The first inequality in (3.28) follows from (POS), whereas the second is proved analogously to the proof of Theorem 3.4, using (3.27) and the definition of σ_n . Hence, pasting at the stopping time σ is in accordance with Proposition 3.2.

Finally, the downward jumps at τ^* and at σ , together with the zero control in between, ensure that (\bar{Y}, \bar{Z}) satisfies (3.1), as was shown in Step 4c of the proof of Theorem 3.4. The rest of the proof does not need any further alterations. \square

Also the positivity assumption (POS) on the generator can be relaxed to a linear bound below, which however has to be consistent with the assumption (NOR'). In the following we say that a generator g is

(LB-NOR') linearly bounded from below under (NOR'), if there exist adapted measurable processes a and b with values in $\mathbb{R}^{1 \times d}$ and \mathbb{R} , respectively, such that $g(y, z) \geq az^T - b$, for all $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$, and

$$\frac{dP^a}{dP} = \mathcal{E} \left(\int_0^\cdot a dW \right)_T \quad (3.29)$$

defines an equivalent probability measure P^a . Furthermore, $\int_0^t b_s ds \in L^1(P^a)$ holds for all $t \in [0, T]$, and a and b are such that the positive generator defined by

$$\bar{g}(y, z) := g \left(y + \int_0^\cdot b_s ds, z \right) - az^T - b, \quad \text{for all } (y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}, \quad (3.30)$$

satisfies (NOR').

An (LB-NOR') setting can always be reduced to a setting with generator satisfying (POS) and (NOR'), by using the change of measure (3.29) and \bar{g} defined in (3.30). Hence, Lemma 3.1 and Proposition 3.2, which strongly rely on the property (POS), can be applied. Note that for the case $b = 0$, the generator \bar{g} even satisfies (POS) and (NOR). However, we need a slightly different definition of admissibility than before. A control process Z is said to be a -admissible, if $\int Z dW^a$ is a P^a -supermartingale, where $W^a = (W^1 - \int a^1 ds, \dots, W^d - \int a^d ds)^T$ is a P^a -Brownian motion by Girsanov's theorem.

The set $\mathcal{A}^a(\xi, g) := \{(Y, Z) \in \mathcal{S} \times \mathcal{L} : Z \text{ is } a\text{-admissible and (3.1) holds}\}$, as well as the process

$$\mathcal{E}_t^{g,a}(\xi) = \text{ess inf}\{Y_t \in L^0(\mathcal{F}_t) : (Y, Z) \in \mathcal{A}^a(\xi, g)\}, \quad \text{for } t \in [0, T],$$

are defined analogously to (3.2) and (3.3), respectively. We are now ready to state our most general result, which follows from Corollary 3.7 and [6, Theorem 4.16].

Theorem 3.8. *Let g be a generator satisfying (LB-NOR') and $\xi \in L^0(\mathcal{F}_T)$ a terminal condition such that $\xi^- \in L^1(P^a)$. If in addition $\mathcal{A}^a(\xi, g) \neq \emptyset$, then there exists a unique a -admissible control \hat{Z} such that $(\mathcal{E}^{g,a}(\xi), \hat{Z}) \in \mathcal{A}^a(\xi, g)$.*

3.4 Continuous Local Martingales and Controls in \mathcal{L}^1

Under stronger integrability conditions, the techniques used in the proof of Theorem 3.4 can be generalized to the case where the Brownian motion W appearing in the stochastic integral in (3.1) is replaced by a d -dimensional continuous local martingale M . Let us assume that M is adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$, which satisfies the usual conditions and in which all martingales are continuous and all stopping times are predictable. We consider controls within the set $\mathcal{L}^1 := \mathcal{L}^1(M)$, consisting of all $\mathbb{R}^{1 \times d}$ -valued, progressively measurable processes Z , such that $\int Z dM \in \mathcal{H}^1$. As before, for $Z \in \mathcal{L}^1$ the stochastic integral $(\int_0^t Z_s dM_s)_{t \in [0, T]}$ is well defined and is by means of the Burkholder-Davis-Gundy inequality a continuous martingale. A pair $(Y, Z) \in \mathcal{S} \times \mathcal{L}^1$ is now called a supersolution of a BSDE, if it satisfies, for $0 \leq s \leq t \leq T$,

$$Y_s - \int_s^t g_u(Y_u, Z_u) d\langle M \rangle_u + \int_s^t Z_u dM_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi, \quad (3.31)$$

for a normal integrand g as generator and a terminal condition $\xi \in L^0(\mathcal{F}_T)$. We will focus on the set

$$\mathcal{A}^{M,1}(\xi, g) := \{(Y, Z) \in \mathcal{S} \times \mathcal{L}^1 : (Y, Z) \text{ satisfy (3.31)}\}.$$

If we assume $\mathcal{A}^{M,1}(\xi, g)$ to be non-empty, Theorem 3.4 combined with compactness results for sequences of \mathcal{H}^1 -bounded martingales given in Delbaen and Schachermayer [3] yields that

$$\mathcal{E}_t^g(\xi) := \text{ess inf}\{Y_t \in L^0(\mathcal{F}_t) : (Y, Z) \in \mathcal{A}^{M,1}(\xi, g)\}, \quad t \in [0, T], \quad (3.32)$$

is the value process of the unique minimal supersolution within $\mathcal{A}^{M,1}(\xi, g)$. Note that Lemma 3.1 and Proposition 3.2 extend to the case where W is substituted by M .

Theorem 3.9. *Assume that the generator g satisfies (POS) and (NOR) and let $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $(E[\xi^- | \mathcal{F}_t])^* \in L^1(\mathcal{F}_T)$. If $\mathcal{A}^{M,1}(\xi, g) \neq \emptyset$, then there exists a unique \hat{Z} such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}^{M,1}(\xi, g)$.*

Proof. By assumption, there is some $(Y^b, Z^b) \in \mathcal{A}^{M,1}(\xi, g)$ and we consider, without loss of generality, only those pairs $(Y, Z) \in \mathcal{A}^{M,1}(\xi, g)$ satisfying $Y \leq Y^b$, obtained by suitable pasting as in Proposition 3.2. Using the techniques of the proof of Theorem 3.4, we can find a sequence $((Y^n, Z^n)) \subset \mathcal{A}^{M,1}(\xi, g)$ satisfying $\lim_n \|Y^n - \mathcal{E}^{g,+}(\xi)\|_{\mathcal{R}^\infty} = 0$, in analogy to (3.6). Since $(\int Z^n dM)$ is uniformly bounded in \mathcal{H}^1 , compare [6, Theorem 4.5], it follows from [1, Theorem 1] that $\mathcal{E}^{g,+}(\xi)$ is a special semimartingale with canonical decomposition $\mathcal{E}^{g,+}(\xi) = \mathcal{E}_0^{g,+}(\xi) + N - A$ and that

$$\lim_{n \rightarrow \infty} \left\| \int Z^n dM - N \right\|_{\mathcal{H}^1} = 0. \quad (3.33)$$

Moreover, $N \in \mathcal{H}^1$. Now [3, Theorem 1.6] yields the existence of some $\hat{Z} \in \mathcal{L}^1$ such that $N = \int \hat{Z} dM$. By means of (3.33), (Z^n) converges, up to a subsequence, $P \otimes d\langle M \rangle_t$ -almost surely to \hat{Z} and $\lim_n \int_0^t Z^n dM = \int_0^t \hat{Z} dM$, for all $t \in [0, T]$, P -almost surely, by means of the Burkholder-Davis-Gundy inequality. In particular, $\lim_{n \rightarrow \infty} Z^n(\omega) = \hat{Z}(\omega)$, $d\langle M \rangle_t$ -almost surely, for almost all $\omega \in \Omega$. Verifying that $(\mathcal{E}^{g,+}(\xi), \hat{Z})$ satisfy (3.31) is now done analogously to Step 1 in the proof of Theorem 3.4, and hence we are done. \square

References

- [1] M. Barlow and P. E. Protter. On Convergence of Semimartingales. *Séminaire de Probabilités XXIV, Lect. Notes Math. 1426*, pages 188–193, 1990.
- [2] P. Cheridito and M. Stadje. Existence, Minimality and Approximation of Solutions to BSDEs with Convex Drivers. *ArXiv e-prints*, 2011.
- [3] F. Delbaen and W. Schachermayer. A Compactness Principle for Bounded Sequences of Martingales with Applications. *Proceedings of the Seminar of Stochastic Analysis, Random Fields and Applications, Progress in Probability*, pages 133–173, Birkhäuser, 1996.
- [4] F. Delbaen, Y. Hu, and X. Bao. Backward SDEs with Superquadratic Growth. *Probability Theory and Related Fields*, 150(1-2):145–192, 2011.
- [5] C. Dellacherie and P. A. Meyer. *Probabilities and Potential. B*, volume 72 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1982.
- [6] S. Drapeau, G. Heyne, and M. Kupper. Minimal Supersolutions of Convex BSDEs. *ArXiv e-prints*, 2011.
- [7] N. El Karoui, S. Peng, and M. C. Quenez. Backward Stochastic Differential Equation in Finance. *Mathematical Finance*, 1(1):1–71, 1997.
- [8] O. Kallenberg. *Foundations of Modern Probability*. Probability and its Applications (New York). Springer-Verlag, New York, 2nd edition, 2002.
- [9] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2nd edition, 1991.
- [10] M. Kobylanski. Backward Stochastic Differential Equations and Partial Differential Equations with Quadratic Growth. *Annals of Probability*, 28(2):558–602, 2000.
- [11] E. Pardoux and S. Peng. Adapted Solution of a Backward Stochastic Differential Equation. *System & Control Letters*, 14(1):55–61, 1990.
- [12] S. Peng. Backward SDE and related g-expectation. *Backward Stochastic Differential Equation, Pitman Research Notes in Mathematics Series 364*, Longman, Harlow, pages 141–159, 1997.
- [13] S. Peng. Monotonic Limit Theorem of BSDE and Nonlinear Decomposition Theorem of Doob-Meyer’s Type. *Probability Theory and Related Fields*, 113(4):473–499, 1999.

- [14] S. Peng. G-Expectation, G-Brownian Motion and Related Stochastic Calculus of Itô Type. *Stochastic Analysis and Applications*, volume 2 of Abel Symp.:541–567, 2007.
- [15] S. Peng. Multi-Dimensional G-Brownian Motion and Related Stochastic Calculus under G-Expectation. *Stochastic Processes and Their Applications*, 12:2223–2253, 2008.
- [16] P. E. Protter. *Stochastic Integration and Differential Equations*. Springer, 2nd edition, 2005.
- [17] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*, volume 293 of *Fundamental Principles of Mathematical Sciences*. Springer-Verlag, Berlin, 3rd edition, 1999.
- [18] R. T. Rockafellar. Integral Functionals, Normal Integrands and Measurable Selections. *Nonlinear Operators and the Calculus of Variations*, 543 of *Lecture Notes in Mathematics*:157–207, 1976.
- [19] R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis*. Springer, Berlin, New York, 1998.
- [20] H. M. Soner and N. Touzi. Stochastic Target Problems, Dynamic Programming, and Viscosity Solutions. *SIAM Journal on Control Optimization*, 41(2):404–424, 2002.

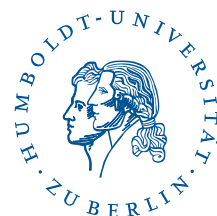
SFB 649 Discussion Paper Series 2011

For a complete list of Discussion Papers published by the SFB 649, please visit <http://sfb649.wiwi.hu-berlin.de>.

- 001 "Localising temperature risk" by Wolfgang Karl Härdle, Brenda López Cabrera, Ostap Okhrin and Weining Wang, January 2011.
- 002 "A Confidence Corridor for Sparse Longitudinal Data Curves" by Shuzhuan Zheng, Lijian Yang and Wolfgang Karl Härdle, January 2011.
- 003 "Mean Volatility Regressions" by Lu Lin, Feng Li, Lixing Zhu and Wolfgang Karl Härdle, January 2011.
- 004 "A Confidence Corridor for Expectile Functions" by Esra Akdeniz Duran, Mengmeng Guo and Wolfgang Karl Härdle, January 2011.
- 005 "Local Quantile Regression" by Wolfgang Karl Härdle, Vladimir Spokoiny and Weining Wang, January 2011.
- 006 "Sticky Information and Determinacy" by Alexander Meyer-Gohde, January 2011.
- 007 "Mean-Variance Cointegration and the Expectations Hypothesis" by Till Strohsal and Enzo Weber, February 2011.
- 008 "Monetary Policy, Trend Inflation and Inflation Persistence" by Fang Yao, February 2011.
- 009 "Exclusion in the All-Pay Auction: An Experimental Investigation" by Dietmar Fehr and Julia Schmid, February 2011.
- 010 "Unwillingness to Pay for Privacy: A Field Experiment" by Alastair R. Beresford, Dorothea Kübler and Sören Preibusch, February 2011.
- 011 "Human Capital Formation on Skill-Specific Labor Markets" by Runli Xie, February 2011.
- 012 "A strategic mediator who is biased into the same direction as the expert can improve information transmission" by Lydia Mechtenberg and Johannes Münster, March 2011.
- 013 "Spatial Risk Premium on Weather Derivatives and Hedging Weather Exposure in Electricity" by Wolfgang Karl Härdle and Maria Osipenko, March 2011.
- 014 "Difference based Ridge and Liu type Estimators in Semiparametric Regression Models" by Esra Akdeniz Duran, Wolfgang Karl Härdle and Maria Osipenko, March 2011.
- 015 "Short-Term Herding of Institutional Traders: New Evidence from the German Stock Market" by Stephanie Kremer and Dieter Nautz, March 2011.
- 016 "Oracally Efficient Two-Step Estimation of Generalized Additive Model" by Rong Liu, Lijian Yang and Wolfgang Karl Härdle, March 2011.
- 017 "The Law of Attraction: Bilateral Search and Horizontal Heterogeneity" by Dirk Hofmann and Salmai Qari, March 2011.
- 018 "Can crop yield risk be globally diversified?" by Xiaoliang Liu, Wei Xu and Martin Odening, March 2011.
- 019 "What Drives the Relationship Between Inflation and Price Dispersion? Market Power vs. Price Rigidity" by Sascha Becker, March 2011.
- 020 "How Computational Statistics Became the Backbone of Modern Data Science" by James E. Gentle, Wolfgang Härdle and Yuichi Mori, May 2011.
- 021 "Customer Reactions in Out-of-Stock Situations – Do promotion-induced phantom positions alleviate the similarity substitution hypothesis?" by Jana Luisa Diels and Nicole Wiebach, May 2011.

SFB 649, Spandauer Str. 1, D-10178 Berlin
<http://sfb649.wiwi.hu-berlin.de>

This research was supported by the Deutsche
Forschungsgemeinschaft through the SFB 649 "Economic Risk".



SFB 649 Discussion Paper Series 2011

For a complete list of Discussion Papers published by the SFB 649, please visit <http://sfb649.wiwi.hu-berlin.de>.

- 022 "Extreme value models in a conditional duration intensity framework" by Rodrigo Herrera and Bernhard Schipp, May 2011.
- 023 "Forecasting Corporate Distress in the Asian and Pacific Region" by Russ Moro, Wolfgang Härdle, Saeideh Aliakbari and Linda Hoffmann, May 2011.
- 024 "Identifying the Effect of Temporal Work Flexibility on Parental Time with Children" by Juliane Scheffel, May 2011.
- 025 "How do Unusual Working Schedules Affect Social Life?" by Juliane Scheffel, May 2011.
- 026 "Compensation of Unusual Working Schedules" by Juliane Scheffel, May 2011.
- 027 "Estimation of the characteristics of a Lévy process observed at arbitrary frequency" by Johanna Kappus and Markus Reiß, May 2011.
- 028 "Asymptotic equivalence and sufficiency for volatility estimation under microstructure noise" by Markus Reiß, May 2011.
- 029 "Pointwise adaptive estimation for quantile regression" by Markus Reiß, Yves Rozenholc and Charles A. Cuenod, May 2011.
- 030 "Developing web-based tools for the teaching of statistics: Our Wikis and the German Wikipedia" by Sigbert Klinke, May 2011.
- 031 "What Explains the German Labor Market Miracle in the Great Recession?" by Michael C. Burda and Jennifer Hunt, June 2011.
- 032 "The information content of central bank interest rate projections: Evidence from New Zealand" by Gunda-Alexandra Detmers and Dieter Nautz, June 2011.
- 033 "Asymptotics of Asynchronicity" by Markus Bibinger, June 2011.
- 034 "An estimator for the quadratic covariation of asynchronously observed Itô processes with noise: Asymptotic distribution theory" by Markus Bibinger, June 2011.
- 035 "The economics of TARGET2 balances" by Ulrich Bindseil and Philipp Johann König, June 2011.
- 036 "An Indicator for National Systems of Innovation - Methodology and Application to 17 Industrialized Countries" by Heike Belitz, Marius Clemens, Christian von Hirschhausen, Jens Schmidt-Ehmcke, Axel Werwatz and Petra Zloczynski, June 2011.
- 037 "Neurobiology of value integration: When value impacts valuation" by Soyoung Q. Park, Thorsten Kahnt, Jörg Rieskamp and Hauke R. Heekeren, June 2011.
- 038 "The Neural Basis of Following Advice" by Guido Biele, Jörg Rieskamp, Lea K. Krugel and Hauke R. Heekeren, June 2011.
- 039 "The Persistence of "Bad" Precedents and the Need for Communication: A Coordination Experiment" by Dietmar Fehr, June 2011.
- 040 "News-driven Business Cycles in SVARs" by Patrick Bunk, July 2011.
- 041 "The Basel III framework for liquidity standards and monetary policy implementation" by Ulrich Bindseil and Jeroen Lamoot, July 2011.
- 042 "Pollution permits, Strategic Trading and Dynamic Technology Adoption" by Santiago Moreno-Bromberg and Luca Taschini, July 2011.
- 043 "CRRA Utility Maximization under Risk Constraints" by Santiago Moreno-Bromberg, Traian A. Pirvu and Anthony Réveillac, July 2011.

SFB 649, Spandauer Str. 1, D-10178 Berlin
<http://sfb649.wiwi.hu-berlin.de>

This research was supported by the Deutsche
Forschungsgemeinschaft through the SFB 649 "Economic Risk".



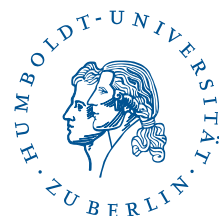
SFB 649 Discussion Paper Series 2011

For a complete list of Discussion Papers published by the SFB 649, please visit <http://sfb649.wiwi.hu-berlin.de>.

- 044 "Predicting Bid-Ask Spreads Using Long Memory Autoregressive Conditional Poisson Models" by Axel Groß-Klußmann and Nikolaus Hautsch, July 2011.
- 045 "Bayesian Networks and Sex-related Homicides" by Stephan Stahlschmidt, Helmut Tausendteufel and Wolfgang K. Härdle, July 2011.
- 046 "The Regulation of Interdependent Markets", by Raffaele Fiocco and Carlo Scarpa, July 2011.
- 047 "Bargaining and Collusion in a Regulatory Model", by Raffaele Fiocco and Mario Gilli, July 2011.
- 048 "Large Vector Auto Regressions", by Song Song and Peter J. Bickel, August 2011.
- 049 "Monetary Policy, Determinacy, and the Natural Rate Hypothesis", by Alexander Meyer-Gohde, August 2011.
- 050 "The impact of context and promotion on consumer responses and preferences in out-of-stock situations", by Nicole Wiebach and Jana L. Diels, August 2011.
- 051 "A Network Model of Financial System Resilience", by Kartik Anand, Prasanna Gai, Sujit Kapadia, Simon Brennan and Matthew Willison, August 2011.
- 052 "Rollover risk, network structure and systemic financial crises", by Kartik Anand, Prasanna Gai and Matteo Marsili, August 2011.
- 053 "When to Cross the Spread: Curve Following with Singular Control" by Felix Naujokat and Ulrich Horst, August 2011.
- 054 "TVICA - Time Varying Independent Component Analysis and Its Application to Financial Data" by Ray-Bing Chen, Ying Chen and Wolfgang K. Härdle, August 2011.
- 055 "Pricing Chinese rain: a multi-site multi-period equilibrium pricing model for rainfall derivatives" by Wolfgang K. Härdle and Maria Osipenko, August 2011.
- 056 "Limit Order Flow, Market Impact and Optimal Order Sizes: Evidence from NASDAQ TotalView-ITCH Data" by Nikolaus Hautsch and Ruihong Huang, August 2011.
- 057 "Optimal Display of Iceberg Orders" by Gökhan Cebiroğlu and Ulrich Horst, August 2011.
- 058 "Optimal liquidation in dark pools" by Peter Kratz and Torsten Schöneborn, September 2011.
- 059 "The Merit of High-Frequency Data in Portfolio Allocation" by Nikolaus Hautsch, Lada M. Kyj and Peter Malec, September 2011.
- 060 "On the Continuation of the Great Moderation: New evidence from G7 Countries" by Wenjuan Chen, September 2011.
- 061 "Forward-backward systems for expected utility maximization" by Ulrich Horst, Ying Hu, Peter Imkeller, Anthony Réveillac and Jianing Zhang.
- 062 "On heterogeneous latent class models with applications to the analysis of rating scores" by Aurélie Bertrand and Christian M. Hafner, October 2011.
- 063 "Multivariate Volatility Modeling of Electricity Futures" by Luc Bauwens, Christian Hafner and Diane Pierret, October 2011.

SFB 649, Spandauer Str. 1, D-10178 Berlin
<http://sfb649.wiwi.hu-berlin.de>

This research was supported by the Deutsche
Forschungsgemeinschaft through the SFB 649 "Economic Risk".



SFB 649 Discussion Paper Series 2011

For a complete list of Discussion Papers published by the SFB 649, please visit <http://sfb649.wiwi.hu-berlin.de>.

- 064 "Semiparametric Estimation with Generated Covariates" by Enno Mammen, Christoph Rothe and Melanie Schienle, October 2011.
- 065 "Linking corporate reputation and shareholder value using the publication of reputation rankings" by Sven Tischer and Lutz Hildebrandt, October 2011.
- 066 "Monitoring, Information Technology and the Labor Share" by Dorothee Schneider, October 2011.
- 067 "Minimal Supersolutions of BSDEs with Lower Semicontinuous Generators" by Gregor Heyne, Michael Kupper and Christoph Mainberger, October 2011.

SFB 649, Spandauer Str. 1, D-10178 Berlin
<http://sfb649.wiwi.hu-berlin.de>

This research was supported by the Deutsche
Forschungsgemeinschaft through the SFB 649 "Economic Risk".

